**Department of Mathematics**

**PhD Qualifying Exam in Analysis (MATH 5453-5463)**

*May 2010*

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**Directions:** Work as many problems as you can; there are 100 total points. If you are asked to "state" a theorem, then no proof is expected unless it is asked for explicitly. You have three hours to complete this exam.

**Notation:** Unless specified otherwise, $L^p(\mu)$ (for $1 \le p \le \infty$) denotes the space of $L^p$ functions on an abstract measure space $(X, \mathcal{M}, \mu)$. We use notations such as $L^p([a,b])$, $L^p([a,\infty)$, and $L^p(\mathbb{R}))$ to denote the space of $L^p$ functions on the indicated interval with respect to the standard Lebesgue measure, which we denote by $\mathfrak{m}$. Integrals with respect to the standard Lebesgue measure may be denoted by either $\int_{[a,b]} f\, d\mathfrak{m}$ or $\int_a^b f(t)\, d\mathfrak{m}(t)$. We also use $C([a,b])$ to denote the set of continuous real-valued functions on the interval $[a,b]$.

(1) Consider the metric spaces $\big(C([0,1]), d_1\big)$ and $\big(C([0,1]), d_2\big)$, where the metrics $d_1$ and $d_2$ are defined by

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx, \quad d_2(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}, \quad (\text{for } f, g \in C([0,1])).$$

Also let $I : C([0,1]) \to C([0,1])$ denote the identity mapping $I(f) = f$.

(a) [5pts] Is the mapping $I : \big(C([0,1]), d_1\big) \to \big(C([0,1]), d_2\big)$ continuous? Justify your answer by either a proof (if the assertion is true) or a counterexample (if the assertion is false).

(b) [5pts] Is the mapping $I : \big(C([0,1]), d_2\big) \to \big(C([0,1]), d_1\big)$ continuous? Justify your answer by either a proof (if the assertion is true) or a counterexample (if the assertion is false).

(c) [4pts] One of the metric spaces $\big(C([0,1]), d_1\big)$ and $\big(C([0,1]), d_2\big)$ is complete and the other is not. Which is which? Explain your answer, but you don't have to provide any detailed proofs.

(2) Let $(r_n)$ be an enumeration of the rational numbers $\mathbb{Q}$ and let $A \subseteq \mathbb{R}$ be the set defined by

$$A = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left( r_n - \frac{1}{m2^{n+1}}, r_n + \frac{1}{m2^{n+1}} \right).$$

(a) [3pts] Explain why $A$ is a Lebesgue measurable set and compute its Lebesgue measure $\mathfrak{m}(A)$.

(b) [3pts] With respect to the complete metric space $\mathbb{R}$ (usual metric understood) is $A$ a first category set or a second category set? Explain.

(c) [3pts] Is $A = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \left( r_n - \frac{1}{m2^{n+1}}, r_n + \frac{1}{m2^{n+1}} \right)$? Explain.

(3) (a) [3pts] State Fatou's lemma for the Lebesgue integral.

(b) [8pts] Let $(X, \mathcal{M}, \mu)$ be a measure space, let $1 \le p < \infty$, and let $\{f_n : X \to \mathbb{R} \mid n \in \mathbb{N}\}$ be a sequence of functions in $L^p(\mu)$ such that for some real number $M > 0$ we have $\|f_n\|_p \le M$ for every $n \in \mathbb{N}$. If $(f_n)$ converges in measure to a measurable function $f : X \to \mathbb{R}$, then prove that $f \in L^p(\mu)$.

(c) [3pts] For the situation in part (b) can we also conclude that $f_n \to f$ in the $L^p$ norm? Explain.

(4) [8pts] If $(X, \mathcal{M}, \mu)$ is a measure space and if $f \in L^1(\mu)$, then prove that for every $\epsilon > 0$ there exists a measurable set $F \in \mathcal{M}$ such that $\mu(F) < \infty$ and $\int_{X \setminus F} |f|\, d\mu < \epsilon$.

(5) [8pts] Prove that if $f \in L^1([a,b])$ has the property that $\int_a^x f(t)\, d\mathfrak{m}(t) = 0$ for every $x \in [a,b]$, then $f = 0$ a. e. on $[a,b]$.

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(6) (a) [3pts] State Hölder's inequality.

(b) [5pts] Let $(X, \mathcal{M}, \mu)$ be a measure space and let $r, s \in \mathbb{R}$ be such that $r > 1$ and $s > 1$ (note that $r$ and $s$ are otherwise arbitrary). Find a positive real number $\alpha$ for which the following assertion is true.

$$f \in L^r(\mu) \text{ and } g \in L^s(\mu) \quad \Rightarrow \quad |fg|^\alpha \in L^1(\mu)$$

(your answer will depend on $r$ and $s$).

(7) [8pts] Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous at every point of the closed interval $[a, b]$, differentiable at every point of the open interval $(a, b)$, and for which there exists a real number $M > 0$ such that $|f'(x)| \leq M$ for every $x \in (a, b)$. Prove that $f$ is absolutely continuous on $[a, b]$.

(8) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be $\sigma$-finite measure spaces, and let $(X \times Y, \mathcal{M} \times \mathcal{N}, \mu \times \nu)$ be the product measure space.

(a) [3pts] If $E \in \mathcal{M} \times \mathcal{N}$, then express the product measure $(\mu \times \nu)(E)$ in terms of two equivalent integrals, where one integral is with respect to the measure $\mu$ and the other integral is with respect to the measure $\nu$ (this is part of the statement of the product measure theorem, but no proofs are being asked for here).

(b) [7pts] Prove that if $\lambda : \mathcal{M} \to [0, \infty]$ and $\eta : \mathcal{N} \to [0, \infty]$ are $\sigma$-finite measures for which $\lambda \ll \mu$ and $\eta \ll \nu$, then $\lambda \times \eta \ll \mu \times \nu$.

(9) This problem deals with the "counting measure space" $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where $\mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$ and $\mu$ is the counting measure.

(a) [4pts] Describe the space $L^p(\mu)$ for $1 \leq p < \infty$ (be as specific as possible).

(b) [3pts] Give an example of a function $f : \mathbb{N} \to \mathbb{R}$ such that $f \in L^2(\mu)$, but $f \notin L^1(\mu)$.

(c) [6pts] Prove that $L^1(\mu) \subseteq L^2(\mu)$.

(10) (a) [3pts] State the Riesz Representation Theorem for $L^p(\mu)$.

(b) [5pts] For the counting measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ (see Problem (9)) show that $\Lambda : L^2(\mu) \to \mathbb{R}$ defined by

$$\Lambda(f) = \sum_{n=1}^{\infty} \frac{f(n)}{n} \quad (f \in L^2(\mu))$$

is a bounded linear functional on $L^2(\mu)$ and compute the norm of $\Lambda$.