Instructons:

1. Please write a neat, clear, thoughtful, and hopefully correct solution to each of the following problems. Please show *all* relevent work.
2. You should do as many problems as the time allows. You are not expected to answer all parts of all questions. *However you should do at least two problems from each of the four sections.*
3. Each problem is worth the same. Partial credit will be given, but a complete solution of one problem is worth more than partial work on two problems.
4. Good luck.

Standing Assumptions:

1. All rings are assumed to have 1.
2. All modules are assumed to be unital left modules.

1 Groups

1. Let X be a set with 5 elements and let G be a group of order 128. Say that X is a G-set (ie. that G acts on the set X). Prove that there is at least two elements $g_1, g_2 \in G$ with the property that $g_1.x = x$ and $g_2.x = x$ for all $x \in X$ (where we write $g.x$ for the action of $g \in G$ on $x \in X$).
2. Let G be a nonabelian finite group. Prove that $|Z(G)| \leq |G|/4$, where $Z(G)$ denotes the center of G. Hint: Think about the factor group $G/Z(G)$.
3. (a) Prove that every group of order 45 is abelian.
   (b) How many (nonisomorphic) groups of order 45 are there? Write down exactly one group from each isomorphism class.
4. How many elements of order 5 are there in a simple group of order 120?

2 Rings

1. Find the greatest common divisor of the polynomials $p(x) = x^4 + x^3 + 2x^2 + x + 1$ and $q(x) = x^5 + 2x^3 + x$ in $\mathbb{R}[x]$.
2. In the ring $\mathbb{Z}[x]$, consider the ideal I generated by the elements 2 and $x^2 + x + 1$.
   (a) How many elements does the quotient ring $\mathbb{Z}[x]/I$ contain? Please justify your answer.
   (b) Is the ideal I a maximal ideal? Please justify your answer.
3. Let R be a commutative ring with 1. Let J be the intersection of all maximal ideals of R. Show that $x \in J$ if and only if for every $y \in R$, $1 - xy$ is a unit in R.
4. Let R be a not necessarily commutative ring with 1. We say R is a *simple* ring if it has no nontrivial ideals. Let
$$Z = \{r \in R \mid rx = xr \text{ for all } x \in R\}.$$
   If R is a simple ring, please show Z is necessarily a field.

3 Modules

1. If M is a finitely generated module over a Noetherian ring and $f : M \to M$ is an epimorphism, prove that f is an automorphism.

2. If R is a commutative ring and Q is an R-module, then we call Q *injective* if it has the property that whenever you have R-modules M and N, an R-module homomorphism $f : M \to Q$, and an *injective* R-module homomorphism $\varphi : M \to N$, then you have an R-module homomorphism $\bar{f} : N \to Q$ such that
$$\bar{f} \circ \varphi = f.$$
Let F be a field and let Q be a finite dimensional vector space over F (aka an F-module). Please show Q is an injective F-module. It may be helpful to draw the commutative diagram to better understand the definition of injective.

3. (a) Let CommRings and Rings be the categories of Commutative Rings and Rings, respectively. Let
$$\mathcal{M} : \text{CommRings} \to \text{Rings}$$
be given by
$$\mathcal{M}(R) = M_2(R),$$
where $M_2(R)$ is all 2×2 matrices with entries in the ring R with the usual addition and multiplication of matricies. Please define $\mathcal{M}$ on homomorphisms in such a way as to make $\mathcal{M}$ into a functor. Please verify that $\mathcal{M}$ is a functor.

   (b) Let R be a commutative ring. Let R-mod and $M_2(R)$-mod be the categories of all R-modules and all $M_2(R)$-modules respectively. Let
$$\mathcal{C} : R\text{-mod} \to M_2(R)\text{-mod},$$
be given by
$$\mathcal{C}(M) = M_{2\times 1}(M)$$
where $M_{2\times 1}(M)$ is all column vectors of height 2 with entries from M. Please define $\mathcal{C}$ on morphisms in such a way as to obtain a functor. Please verify $\mathcal{C}$ is a functor.

4 Fields

1. Let p be an odd prime. Let F be splitting field of $x^p - 1$ over $\mathbb{Q}$. Prove that there is a unique field K between $\mathbb{Q}$ and F which is of degree 2 over $\mathbb{Q}$.

2. Let $k = \mathbb{Q}(\pi) \subseteq \mathbb{R}$. Set $f(x) = x^3 - \pi \in k[x]$. Let $E \subseteq \mathbb{C}$ be a splitting field for f over k. Use Galois theory to find all the intermediate fields F with $k \subseteq F \subseteq E$.

3. Let $\alpha \in \mathbb{C}$ be a root of $x^3 + 2x + 2$. Please express $(\alpha^2 + 1)^{-1}$ as a polynomial in α.

4. Let E be the algebraic closure of $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ and let $\alpha \in E \backslash \mathbb{F}_3$ be a root of $x^3 + 2x + 2$. Please compute the number of elements in $\mathbb{F}_3(\alpha)$. Be sure to explain your answer.