Topology Qualifying Review Exam

August 20, 2010

Name:

- Prove all the statements unless specified otherwise.

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1. Write definitions. Do as the example.

Example: **quotient topology**

Answer: Let X be a topological space, Y be a set, and $q : X \to Y$ be a map. The strongest topology $\mathcal{T}$ on Y that makes q continuous is called the quotient topology on Y for the map q. In fact, $\mathcal{T}$ is constructed as follows: For $V \subset Y$, $V \in \mathcal{T}$ if and only if $q^{-1}(V)$ is open in X.

(1) partition of unity on X dominated by an open covering $\mathcal{U}$
(2) uniform metric on Y^X
(3) compact-open topology
(4) f is homotopic to g relative to A
(5) Let X, Y be a spaces; $A \subset X$; $f : A \to Y$ a map. The space Z is obtained by attaching X to Y along A via f. Explain the set and topology of Z.

2. Consider the product space $X \times Y$, where Y is compact. Let N be an open set of $X \times Y$ contiaing the slice $x_0 \times Y$, then there exists a "tube" $W \times Y$ such that W is an open subset of X and $x_0 \times Y \subset W \times Y \subset N$.

3. Let D be the closed unit disk embedded in $\mathbb{R}^2$ (as a subspace). Identify all the boundary points of D to a single point in order to form a quotient set Q. Endow Q with the quotient topology. Prove that Q is homeomorphic to the standard 2-sphere S^2 (embedded in $\mathbb{R}^3$ as a subspace).

4. Let X be the product space $X = \Pi_{\alpha \in J} X_\alpha$; $\pi_\alpha : X \to X_\alpha$ the projection. Let $\{\mathbf{x}_n : n \in \mathbb{Z}^+\}$ be a sequence of points in X. Prove: $\{\mathbf{x}_n\} \to \mathbf{x}$ if and only if $\{\pi_\alpha(\mathbf{x}_n)\} \to \pi_\alpha(\mathbf{x})$ for each $\alpha \in J$.

5. Let X be a paracompact Hausdorff space; let $\mathcal{U} = \{U_\alpha : \alpha \in J\}$ be an indexed open covering of X. Prove that there exists a prtition of unity on X dominated by $\mathcal{U}$. [You may assume that $\mathcal{U}$ has a precise refinement covering $\mathcal{V} = \{V_\alpha : \alpha \in J\}$ such that $\bar{V}_\alpha \subset U_\alpha$ for each $\alpha \in J$ by paracompactness].

6. Let X be a locally compact Hausdorff space; let the space of continuous functions $\mathcal{C}(X,Y)$ have the compact-open topology. Prove the "evaluation map"

$$e : X \times \mathcal{C}(X,Y) \to Y$$

defined by $e(x,f) = f(x)$ is continuous. Emphasize where the conditions for X are used.

7. Let $\mathbb{R}^* = \mathbb{R} - \{0\}$, the multiplicative group. It acts on $X = \mathbb{R}^3 - \{\mathbf{0}\}$ by scalar multiplication. More precisely, for $\alpha \in \mathbb{R}^*$ and $\mathbf{x} \in X$,

$$\alpha \cdot \mathbf{x} = \alpha \mathbf{x}$$

7A. Prove that this gives an *action* of $\mathbb{R}^*$ on X.
7B. Find the *stabilizer* (= isotropy subgroup) of this action at $n = (0, 0, 1)$.
7C. What is the orbit space $\mathbb{R}^* \backslash X$?

8. Let B a be connected, path-connected, locally path-connected and semi-locally simply connected topological space. Let $b_0 \in B$. Given a subgroup H of $\pi_1(B, b_0)$, construct a covering space $p : (E, e_0) \to (B, b_0)$ such that $p_*(\pi_1(E, e_0)) = H$.
[Write all the construction. Sketch necessary facts without excessive details].

9. Let $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$ be a covering map. Prove that the deck transformation group is isomorphic to $N_{\pi_1(X,x_0)}\Big(p_*(\pi_1(\hat{X}, \hat{x}_0))\Big)$, the normalizer of $p_*(\pi_1(\hat{X}, \hat{x}_0))$ in $\pi_1(X, x_0)$.

10. Use the van Kampen theorem to calculate the fundamental group of the connected sum $T\#K$, where T is a torus, and K is a Klein bottle.