Attempt all questions.

Q1. (a) Define what it means for a topological space to be *compact* .

(b) Define what it means for a topological space to be *Hausdorff*.

(c) What can you conclude about a continuous bijection from a compact space to a Hausdorff space? (no proof necessary)

(d) Give the definition of the *quotient topology*.

(e) The group $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by (vector) translations. Give a detailed proof that the space $\mathbb{R}^2/\mathbb{Z}^2$ with the quotient topology obtained from the standard topology on $\mathbb{R}^2$ is homeomorphic to the product space $S^1 \times S^1$. Here $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ inherits the subspace topology from the standard topology on $\mathbb{R}^2$. Identify (by name or brief statements) the results that you use in your proof.

Q2. (a) Define what it means for a topological space to be *locally compact*.

(b) Let X be a locally compact, Hausdorff space. Define the *one point compactification* $\widehat{X}$ of X. Describe the topology on $\widehat{X}$ in detail (you do not have to prove that it is a topology).

(c) Prove that the topology described above on $\widehat{X}$ is compact.

(d) Prove that the subspace topology on X inherited from the topology on $\widehat{X}$ agrees with the original topology on X.

(e) Give a proof of or provide a counterexample to the following statement.

"If X and Y are path-connected, locally compact, Hausdorff spaces such that $\pi_1(X) = \pi_1(Y)$, then $\pi_1(\widehat{X}) = \pi_1(\widehat{Y})$ where $\widehat{X}$ (resp. $\widehat{Y}$) is the one-point compactification of X (resp. Y)."

Q3. [True/False] Please supply short reasons for your answers. Either name a theorem/result or provide a counterexample as appropriate.

(a) A quotient space of a path-connected space is path-connected.

(b) A quotient space of a simply connected space is simply connected.

(c) The subspace topology on the Hawaiian Earring, $\bigcup_{n=1}^{\infty}\{(x,y) \mid (x-\frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$, inherited from the standard topology on $\mathbb{R}^2$ is finer than the CW topology on $\bigvee_{n=1}^{\infty} S^1$.

(d) $\overline{S_\Omega}$ is not metrizable because it is not first countable.

(e) $\mathbb{R}^\omega$ with the product topology is path connected.

(f) Every second countable, T_3 space embeds into $\mathbb{R}^\omega$ (with the product topology).

(g) A metric space is compact if and only if it is sequentially compact.

Q4. Let H be the subgroup of $F_2 = F_{\{a,b\}}$ generated by $\{a^2, b^2, ba^2b\}$.

(a) Draw the based covering space, $(\widehat{X}, \widehat{x}_0) \to (X, x_0)$, of the wedge of circles, $X = S^1_a \vee S^1_b$, corresponding to the subgroup $H < F_{\{a,b\}}$.

(b) What is the index $[F_{\{a,b\}} : H]$?

(c) What is the rank of H?

(d) Is $b^3a^2 \in H$?

(e) Determine the group $\mathrm{Aut}(\widehat{X} \to X)$.

(f) Determine the group $N_{F_{\{a,b\}}}(H)$.

(g) Draw the covering space of $X = S^1_a \vee S^1_b$ corresponding to $N_{F_{\{a,b\}}}(H)$.

(h) Is $N_{F_{\{a,b\}}}(H) \lhd F_{\{a,b\}}$?

Q5. Use covering spaces to describe the kernel of the homomorphism

$$\phi : \langle a \,|\, a^2 \rangle * \langle b \,|\, b^5 \rangle \;\to\; \langle a \,|\, a^2 \rangle \times \langle b \,|\, b^5 \rangle$$

defined by $a \mapsto a$, $b \mapsto b$. Follow the steps below.

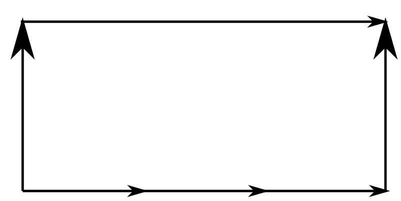
(a) Construct a presentation 2–complex, X, for $\langle a \,|\, a^2 \rangle * \langle b \,|\, b^5 \rangle$ and a presentation 2–complex, Y, for $\langle a \,|\, a^2 \rangle \times \langle b \,|\, b^5 \rangle$. Do this in such a way that the inclusion $i : X \hookrightarrow Y$ induces the homomorphism ϕ.

(b) Describe the universal covering space $\widetilde{Y} \to Y$, and use this to determine a covering space $\widehat{X} \to X$ corresponding to $\ker(\phi)$.

(c) Describe $\ker(\phi)$ as an abstract group.

(d) Describe an explicit set of generators for $\ker(\phi)$ (i.e., as a subgroup of $\mathbb{Z}_2 * \mathbb{Z}_5$).

Q6. (a) Let X be the cell complex obtained by attaching the following 2–cell to the wedge of two circles, $S^1_a \vee S^1_t$. Use van Kampen's theorem to compute $\pi_1(X)$.



(b) Describe the universal covering space $\widetilde{X}$ of X.

(c) Use $\widetilde{X}$ to prove that $\pi_1(X)$ is not an abelian group.

(d) Is there a retraction from X to S^1_t? Construct an explicit retraction (quoting whatever theorems from general topology that may be necessary to perform the construction), or prove that none exists.

(e) Is there a retraction from X to S^1_a? Construct an explicit retraction (quoting whatever theorems from general topology that may be necessary to perform the construction), or prove that none exists.