University of Oklahoma Department of Mathematics
Real Analysis Qualifier Exam
August 17, 2011

Directions: Answer each question on a separate page, writing your ID number (not your name) in the upper right corner of each page and the problem number in the upper left corner of each page. Completely justify your work and state which theorems or results you are citing. You have 3 hours to complete this exam.

Lebesgue measure is denoted by m, the real line is denoted by $\mathbb{R}$.

1. Prove that if f is continuous function on $\mathbb{R}$, then the pre-image of a Borel set is Borel.

2. Let m be Lebesgue measure on $[0,1]$. Construct an open subset E of $[0,1]$ such that
   - E is dense in $[0,1]$;
   - $0 \le a < b \le 1$ implies $m(a,b) \cup E > 0$; and
   - $m(E) < 1$.

3. Prove the Simple Approximation Lemma for Lebesgue measure on $\mathbb{R}$: Let f be a bounded measurable function on $\mathbb{R}$. For any $\epsilon > 0$, there exist simple functions ϕ and ψ such that
$$\phi \le f \le \psi \quad \text{and} \quad 0 \le \psi - \phi < \epsilon \text{ on } \mathbb{R}.$$

4. Prove carefully that the collection of all polynomials with rational coefficients is dense in $L^p[0,1]$ for all $1 \le p < \infty$.

5. Prove the following implications for functions on a closed, bounded, nondegenerate interval $[a,b]$:
$$f \text{ Lipschitz} \Rightarrow f \text{ absolutely continuous} \Rightarrow f \text{ bounded variation.}$$
Provide counterexamples to show that none of the implications can be reversed.

6. Let f_0 be continuous on $[0,1]$. Let
$$f_{k+1}(x) = \int_0^x f_k(t)\,\mathrm{d}m(t), \quad k = 1, 2, 3, \dots .$$
Suppose that for every x there is a k, possibly depending on x, such that $f_k(x) = 0$. Prove that f_0 is identically 0 on some nontrivial subinterval of $[0,1]$. (Hint: Baire Category Theorem)

7. Let $\{f_n\}$ be a sequence of measurable functions defined on $\mathbb{R}$ such that $f_n \to f$ pointwise a.e. and suppose that $\int_{\mathbb{R}} |f_n|\,\mathrm{d}m \to \int_{\mathbb{R}} |f|\,\mathrm{d}m < \infty$.
   (a) Show that for each measurable set E, we have
$$\int_E f_n\,\mathrm{d}m \to \int_E f\,\mathrm{d}m.$$
   (b) Is the statement in (a) still true if we require only $\int_{\mathbb{R}} f_n\,\mathrm{d}m \to \int_{\mathbb{R}} f\,\mathrm{d}m$? Prove, or give a counterexample.