Topology Qualifying Exam **Name:**
January 7, 2013

Instructions: Provide justification for each of your answers and make your arguments clear, but try to avoid excessive detail unless they are specifically called for in the problem. Work at least five problems from each part.

PART I

1. Give an example of topological spaces X and Y and a continuous bijection $f : X \to Y$ which is not a homeomorphism. Be sure to prove that f is continuous and that f is not a homeomorphism.

2. Let X and Y be compact Hausdorff spaces. Show that if $f : X \to Y$ is a continuous bijection then f is a homeomorphism.

3. Let X be an infinite set, and let $\mathcal{B}$ be the collection of all subsets $B \subseteq X$ such that $X - B$ has a finite and even number of elements.
(a) Show that $\mathcal{B}$ is a basis for the cofinite topology on X.
(b) Is the conclusion of part (a) true if X is finite? Explain.

4. Use the definition of compactness to show that
(a) a half-open interval $[a, b) \subset \mathbb{R}$ is not compact,
(b) a closed interval $[a, b] \subset \mathbb{R}$ is compact.
(This problem assumes that $\mathbb{R}$ is endowed with the Euclidean topology.)

5. Let $\{x_\alpha \mid \alpha \in J\}$ be a family of topological spaces. Describe the product topology on $\prod_\alpha X_\alpha$ then show
(a) For each $\beta \in J$ the projection map $p_\beta : \prod_\alpha X_\alpha \to X_\beta$ is continuous and a quotient map.
(b) The projection p_β need not be a closed map.

6. (a) Give an outline of a proof (based on definitions) of the Intermediate Value Theorem: *If X is a connected space and $f : X \to \mathbb{R}$ is continuous and there are elements $x_1, x_2 \in X$ with $f(x_1) < 0$ and $f(x_2) > 0$ then there exists $c \in X$ with $f(c) = 0$.*
(b) Part (a) presumes that $\mathbb{R}$ has the Euclidean topology. Is the statement also true if $\mathbb{R}$ has the lower limit topology $\mathbb{R}_\ell$?

7. Prove that metrizable spaces are normal.

PART II

8. Let $S = [-1, 1] \times [-1, 1] = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1 \text{ and } |y| \leq 1\}$ with the Euclidean topology. Find subsets A and B in S satsfying the following properties:
(i) $A \neq \emptyset$ and $B \neq \emptyset$.
(ii) A and B are connected.
(iii) A and B are disjoint.
(iv) The points $(-1, -1)$ and $(1, 1)$ are in A, and the points $(-1, 1)$ and $(1, -1)$ are in B.

9. Let α and α' be homotopic paths from x to y in X, and let β and β' be homotopic paths from y to z in X. Prove that the concatenations $\alpha \cdot \beta$ and $\alpha' \cdot \beta'$ are homotopic.

10. Define what is meant by a retraction from a topological space onto a subspace A. Prove that there does not exist a retraction from the Mobius band to its boundary circle.

11. Describe all connected 3-fold coverings of the wedge of two circles $S^1 \vee S^1$ and indicate which of these are regular coverings. Give justification for your answer.

12. (a) Show that $\mathbb{R}^2 - \mathbb{Q}^2$ is path connected with the Euclidean topology. (Of course, $\mathbb{Q}^2 = \{(x, y) \mid x \in \mathbb{Q} \text{ and } y \in \mathbb{Q}\}$.)
(b) Show that $\mathbb{R}^2 - \mathbb{Q}^2$ is not simply connected.

13. Which of the following are possible? (Justify your answers in each case.)
(a) The covering transformation group of a 6-sheeted covering map $p : X \to Y$ can have 12 elements.
(b) The covering transformation group of a 12-sheeted covering map $p : X \to Y$ can have 6 elements.
(c) The covering transformation group of a 12-sheeted regular covering map $p : X \to Y$ can have 6 elements.

14. The suspension of a topological space X is the quotient space $\Sigma X = X \times [0, 1]/ \sim$ where $(x, t) \sim (y, s)$ if and only if either $(x, t) = (y, s)$ or $s = t = 1$ or $s = t = 0$.
(a) Prove that the suspension ΣX need not be simply connected.
(b) Prove that the suspension ΣX is simply connected if X is connected. (Hint: Use Van Kampen's Theorem.)