Qualifying examination in analysis
Summer 2013

1. Suppose f is an integrable function on $[0, 1]$ and g is a continuous function on $\mathbf{R}$. Define

$$h(t) = \int_0^1 f(x)g(x - t)\, dx$$

for $t \in \mathbf{R}$. Show that h is continuous on $\mathbf{R}$.

2. Suppose f is a bounded continuous function on $[0, \infty)$. Show that

$$\lim_{n\to\infty} \int_0^\infty ne^{-nx} f(x)\, dx = f(0).$$

Hint: change variables in the integral.

3. *(a)* Show that $L^4[0, 1] \subseteq L^3[0, 1]$.

(b) Suppose $\Lambda : L^3[0, 1] \to \mathbf{R}$ is a bounded linear functional. Show that the restriction of Λ to $L^4[0, 1]$ is a bounded linear functional on $L^4[0, 1]$.

(c) Give an example of a function in $L^{4/3}[0, 1]$ which is not in $L^2[0, 1]$.

(d) Give an example (with proof) of a bounded linear functional on $L^3[0, 1]$ which is not the restriction to $L^3[0, 1]$ of a bounded linear functional on $L^2[0, 1]$.

4. Suppose $\{\phi_n\}$ is an orthonormal sequence in $L^2(\mathbf{R})$, and $g \in L^2(\mathbf{R})$. Show that

$$\lim_{n\to\infty} \int_{-\infty}^\infty g(x)\phi_n(x)\, dx = 0.$$

5. Suppose $\{f_n\}$ is a sequence of absolutely continuous functions on $[0, 1]$ such that

(i) for all $n \in \mathbf{N}$, $f_n(0) = 0$.

(ii) for all $n \in \mathbf{N}$ and all $x \in [0, 1]$, $|f_n'(x)| \le 1$.

(iii) the functions $\{f_n'(x)\}$ converge pointwise a.e. on $[0, 1]$ to a limit function $g(x)$, as $n \to \infty$.

Show that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for every $x \in [0, 1]$, and f is absolutely continuous on $[0, 1]$, with $f'(x) = g(x)$ a.e. on $[0, 1]$.

6. Let $X = \{1, 2, 3, 4\}$, and let Σ be the smallest σ-algebra of subsets of X which contains the sets $\{1\}$ and $\{1, 2\}$.

(a) List the sets in Σ.

(b) Give an example of a function $g : X \to \mathbf{R}$ which is not measurable with respect to Σ.

(c) If $f : X \to \mathbf{R}$ is given by $f(x) = (x - 3)(x - 4)$, and μ is a measure on (X, Σ) with $\mu(\{1\}) = 3$ and $\mu(\{1, 2\}) = 8$, find $\int_X f\, d\mu$.

7. Suppose μ and ν are mutually singular measures on a measure space (X, Σ). Suppose λ is a measure on (X, Σ) which is absolutely continuous with respect to μ and absolutely continuous with respect to ν. Show that $\lambda(E) = 0$ for every E in Σ.

8. Suppose (X, Σ, μ) is a σ-finite measure space, and ν is a measure on Σ such that $\nu(E) \leq \mu(E)$ for all $E \in \Sigma$. If f is the Radon-Nikodym derivative of ν with respect to μ, $f = \frac{d\nu}{d\mu}$, show that $f(x) \leq 1$ a.e. (μ) on X.

9. Let λ be Lebesgue measure on $\mathbf{R}$, and let f be a nonnegative measurable function on $\mathbf{R}$. Prove that

$$\int_{-\infty}^{\infty} (f(x))^2 \, dx = \int_0^{\infty} \lambda(\{x : f(x) > t\}) \, 2t \, dt.$$

Hint: first write $\lambda(\{x : f(x) > t\}) = \displaystyle\int_{-\infty}^{\infty} h(x,t) \, dx$, where

$$h(x,t) = \begin{cases} 1 & \text{if } f(x) > t \\ 0 & \text{if } f(x) \leq t. \end{cases}$$