# Algebra Qualifying Exam (January 2014)

Solve as many of the eight problems as you can. It is not necessary to solve everything in order to pass the exam.

1) Prove that if a group $G$ has order $p^e a$, where $p$ is a prime, $e \geq 1$, and $1 \leq a < p$, then $G$ has a proper, normal subgroup, except when $a = e = 1$.

2) Show that every group of order $15$ is cyclic.

3) Let $p$ be an odd prime number. Let $\zeta$ be a primitive $p$-th root of unity in $\mathbb{C}$.

   a) Show that, for each divisor $n$ of $p-1$, the extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ admits a unique intermediate extension of degree $n$ over $\mathbb{Q}$.

   b) Show that the intermediate extension $K$ of $\mathbb{Q}(\zeta)/\mathbb{Q}$ of degree $\frac{p-1}{2}$ is given by $K = \mathbb{Q}(\eta)$, where $\eta = \zeta + \zeta^{p-1}$.

4) Let $E/K$ be a field extension of characteristic $p > 0$. Let $\alpha$ be a root in $E$ of an irreducible polynomial $f(x) = x^p - x - a \in K[x]$.

   a) Prove that $\alpha + 1$ is also a root of $f$.

   b) Prove that the Galois group of $f$ over $K$ is cyclic of order $p$.

5) Let $R$ be the ring $\mathbb{Z}[\sqrt{2}]$.

   a) Give an example for an odd prime number $p$ that is no longer a prime when considered as an element of $R$.

   b) Give an example for an odd prime number $p$ that remains a prime when considered as an element of $R$.

6) Let $V_1, V_2, W_1, W_2$ be vector spaces over a field $K$. Let $f : V_1 \to V_2$ and $g : W_1 \to W_2$ be linear maps. Using the universal property of the tensor product, construct a natural map

$$f \otimes g : V_1 \otimes W_1 \longrightarrow V_2 \otimes W_2.$$

7) Let $K$ be a field.

   a) Let $E$ be an integral domain containing $K$. Assume that $E$, as a $K$-vector space, is finite-dimensional. Show that $E$ is a field.

   b) Let $f \in K[X]$ be an irreducible polynomial, and let $\alpha$ be a root of $f$ in some algebraic closure of $K$. Show, using part a), that $K(\alpha) = K[\alpha]$.

8) Let $R$ be a commutative ring with $1$. Let $I, J$ be ideals in $R$ with $I + J = R$.

   a) Prove that $IJ = I \cap J$.

   b) Prove the Chinese Remainder Theorem: For any pair $a, b \in R$, there exists $x \in R$ with $x \equiv a \bmod I$ and $x \equiv b \bmod J$.

   c) Assume that $IJ = 0$. Prove that $R \cong (R/I) \times (R/J)$.