ANALYSIS QUALIFYING EXAM – FALL 2015

NAME: ....

Complete 5 of the problems below. If you attempt more than 5 questions, then clearly indicate which 5 should be graded.
(Each problem will count for 10 points.)

For full credit you must provide complete arguments, and state domains ranges etc. whenever you introduce functions, variables, sets and so on. You are allowed to (and you should) refer to results we discussed in class – do not reprove basic textbook material – but clearly indicate/cite the results you use.

A note on the notation used below: The symbol $\mathscr{L}^p$ refers to the set of all real-valued functions f such that $|f|^p$ is integrable (the value $p = \infty$ is a bit different). Clearly, the setup implicitly assumes a measure space $(X, \mathscr{F}, \mu)$, so that the domain of f is X, measurable means $\mathscr{F}$–measurable, and integrable means integrable with respect to μ. Sometimes we write $\mathscr{L}^p(0, \infty)$ or alike to explicitly indicate parts of the underlying measure space – in this example $X = (0, \infty)$. As is common practice, integration with respect to the Lebesgue measure, denoted by Leb, is usually written simply as $\int f(x)\, dx$ instead of $\int f(x)\, \mathrm{Leb}(dx)$.

Question	Points
1	
3	
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11	
Total	

Date: August 19, 2015.

1. Define the set E

$$E = \left\{ x \in \mathbb{R} \colon \text{there exists } C > 0 \text{ such that } \left| x - \frac{m}{n} \right| > \frac{C}{|n|^3} \text{ for all } m, n \in \mathbb{Z},\ n \neq 0 \right\}.$$

Compute the Lebesgue measure of $[0,1] \setminus E$.

2. Compute the limit of the sequence of integrals

$$\int_0^{n^2} \frac{x^n}{1 + x^{n+2}} \sin\left(\frac{\pi\, x}{n}\right) dx$$

as $n \to \infty$. Justify all steps in your calculation.

3. Let μ be a finite measure on some measurable space $(X, \mathscr{F})$. Let $F \colon X \to X$ be such that
   - $F^{-1}(A) \in \mathscr{F}$ for all $A \in \mathscr{F}$,
   - if $A \in \mathscr{F}$ with $\mu(A) = 0$, then $\mu(F^{-1}(A)) = 0$.

   Prove the following statements:
   (a) there exists an $h \in \mathscr{L}^1$ such that $\int_A h(x)\, \mu(dx) = \mu(F^{-1}(A))$ for all $A \in \mathscr{F}$. Furthermore, any two such h are equal μ–almost everywhere.
   (b) $\int f(F(x))\, \mu(dx) = \int f(x)\, h(x)\, \mu(dx)$ for every $f \in \mathscr{L}^1$.

4. Consider the Lebesgue measure on the Borel sets of $[0,1]$. For every $1 \leq p \leq \infty$ and every $f \in \mathscr{L}^p$ we denote by $T(f)$ the function $T(f)(x) = \int_0^x f(s)\, ds$.
   (a) Show that for all $1 \leq p \leq \infty$ if $f \in \mathscr{L}^p$, then $T(f) \in \mathscr{L}^p$, i.e. the mapping $T \colon \mathscr{L}^p \to \mathscr{L}^p$ is well-defined.
   (b) Show that for every $1 < p < \infty$ there exists a constant $C_p < 1$ such that $\|\, T(f) \,\|_p \leq C_p \,\|\, f \,\|_p$ for all $f \in \mathscr{L}^p$.
   (c) Show that for $p = 1$ there exists no constant $C_1 < 1$ such that $\|\, T(f) \,\|_1 \leq C_1 \,\|\, f \,\|_1$ for all $f \in \mathscr{L}^1$.
   (d) Show that for $p = 1$ there exists a constant $C_* < 1$ such that $\|\, T^2(f) \,\|_1 \leq C_* \,\|\, f \,\|_1$ for all $f \in \mathscr{L}^1$. ($T^2(f)$ denotes the function obtained by applying T to $T(f)$, i.e. applying T to f twice.)

5. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of continuous non-negative functions on $[0,1]$, that converge pointwise on $[0,1]$ to an integrable function f with $\int_0^1 f(x)\, dx = 0$.
   (a) Suppose that the sequence of integrals $\int_0^1 f_n(x)\, dx$ is bounded. Prove or disprove by a counterexample: $\lim_{n \to \infty} \int_0^1 f_n(x)\, dx = 0$.
   (b) Suppose that the sequence of integrals $\int_0^1 f_n(x)\, dx$ is bounded. Prove or disprove by a counterexample: if $(f_n)_{n \in \mathbb{N}}$ is such that $f_{n+1}(x) \leq f_n(x)$ for all $n \in \mathbb{N}$ and all $x \in [0,1]$, then $\lim_{n \to \infty} \int_0^1 f_n(x)\, dx = 0$.
   (c) Suppose that $f(x) = 0$ for all $x \in [0,1]$. Prove that if $(f_n)_{n \in \mathbb{N}}$ is such that $f_{n+1}(x) \leq f_n(x)$ for all $n \in \mathbb{N}$ and all $x \in [0,1]$, then f_n converges uniformly on $[0,1]$.

6. Let ν_1 and ν_2 be two finite signed measures on the Borel sets of $\mathbb{R}$. Prove that there exists a unique finite signed measure ν on the Borel set of $\mathbb{R}$ such that

$$\int f(x)\, \nu(dx) = \int \Big(\int f(x + y)\, \nu_1(dx) \Big) \nu_2(dy)$$

for all continuous functions $f \colon \mathbb{R} \to \mathbb{R}$ with compact support.

7. Consider the Lebesgue measure on the Borel sets of $(0,\infty)$. Prove that for every $f \in \mathscr{L}^2(0,\infty)$
   (a) the inequality $\left| \int_0^x f(s)\,ds \right|^2 \leq 2\sqrt{x} \int_0^x \sqrt{s}\,|f(s)|^2\,ds$ holds for all $x \in (0,\infty)$
   (b) the inequality $\|F\|_2 \leq 2\,\|f\|_2$, where $F(x) = \frac{1}{x}\int_0^x f(s)\,ds$.

8. Let f be a non-decreasing function on $[a,b]$. Prove that the following two statements are equivalent:
   (i) f is absolutely continuous
   (ii) for every absolutely continuous function g on $[a,b]$
   $$\int_a^x f(s)\,g'(s)\,ds + \int_a^x f'(s)\,g(s)\,ds = f(x)\,g(x) - f(a)\,g(a)$$
   for every $x \in [a,b]$.

9. Let μ be a finite signed measure on the Borel sets of $\mathbb{R}$, and suppose that μ is absolutely continuous with respect to the Lebesgue measure. Prove that for every Borel set $A \subset \mathbb{R}$ the function $t \mapsto \mu\{t + x : x \in A\}$ is continuous in t.

10. Let $(X, \mathscr{F}, \mu)$ be some measure space. Let $f_n, f, g_n, g \colon X \to \mathbb{R}$ for $n \in \mathbb{N}$ be measurable functions. Suppose that $f_n \to f$ and $g_n \to g$ in measure as $n \to \infty$.
    (a) Prove that $f_n + g_n \to f + g$ in measure as $n \to \infty$.
    (b) Assumption that $\mu(X) < \infty$. Prove that $f_n\,g_n \to f\,g$ in measure as $n \to \infty$.
    (c) If $\mu(X) = \infty$, prove or disprove by a counterexample: $f_n\,g_n \to f\,g$ in measure as $n \to \infty$.

11. Let H be a Hilbert space, and denote by $\|\,.\,\|$ its norm and by $\langle\,.,.\,\rangle$ the corresponding inner product. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $H \setminus \{0\}$ such that $\langle x_n - x_m, x_m \rangle = 0$ whenever $n \geq m$. Prove that the sequence $(s_N)_{N\in\mathbb{N}}$ of partial sums $s_N = \sum_{n=0}^{N} \frac{x_n}{\|x_n\|^2}$ converges in H if and only if $\sum_{n=0}^{\infty} \frac{n}{\|x_n\|^2} < \infty$.

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Problem number: . . . .