Real Analysis

Qualifying Examination

Fall 2016

NAME: **I.D. # :**

Complete five (5) of the problems below. If you attempt more than 5 questions, then please clearly indicate which 5 should be graded on this sheet.

1a. Consider the following statement: Let $(X, \mathcal{B}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence in $L^1(\mu)$ that converges uniformly on X to a function $f \in L^1(\mu)$; then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

If the above statement is true prove it. If the above statement is false, (i) show by example and (ii) add a hypothesis to the above statement that the results is a true statement, and give a proof that your modified statement is indeed true.

Let $(X, \mathcal{B}, \mu)$ be a measure space.
1b. State the following theorems.
 (i) Monotone Convergence Theroem
 (ii) Fatou's Lemma
 (iii) Dominated Convergence Thereom

1c. Prove that (ii) implies (iii).

2a. Let f be an increasing real-valued function on $[a, b]$, and

$$E_{u,v} = \{x : D^+ f(x) > u > v > D_- f(x)\},$$

where u and v are rational numbers,

$$D^+ f(x) = \overline{\lim_{h\to 0+}} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad D_- f(x) = \underline{\lim_{h\to 0+}} \frac{f(x) - f(x-h)}{h}$$

Prove that the outer measure $m^*(E_{u,v}) = 0$.

2b. Let f be an increasing, real-valued, differentiable a.e. on $[a, b]$ and the derivative f' is measurable. prove that

$$\int_a^b f'(x)\, dx \leq f(b) - f(a)\, .$$

3a. State Ascoli-Arzelá Theorem $\big($on a metric space $(X, d)\big)$

3b. Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable functions, $n = 1, 2, \ldots$, with $f_n(0) = 0$ and $|f'(0)| \leq 3$ for all n, x . Suppose

$$\lim_{n\to\infty} f_n(x) = g(x)$$

for all x . Prove that $g : \mathbb{R} \to \mathbb{R}$ is continuous.
(Hint: Use the mean value theorem and Ascoli-Arzelá theorem)

4a. Let f be a real-valued twice differentiable function on an open interval (a, b) . Prove that f is convex if and only if $f'' \geq 0$.

4b. Let $\{\alpha_n\}$ be a sequence of nonnegative numbers whose sum is 1 and $\{\xi_n\}$ be a sequence of positive numbers. Then

$$\prod_{n=1}^{\infty} \xi_n^{\alpha_n} \leq \sum_{n=1}^{\infty} \alpha_n \xi_n \, .$$

5a. Define an n-dimensional differentiable manifold.

5b. State the inverse function theorem on differentiable manifolds.

6a. State Orthogonal Projection Theorem in Hilbert Space.

6b. Let M be a subspace of a a Hilbert space V. Let x be in V Prove that if y is in the subspace M, then $(x - y) \perp M$ if and only if y is the unique point in M closest to x, that is, y **is the "best approximation" to** x **in** M

6c. Prove Riesz Representation Theorem in Hilbert Space (not necessarily separable) by using Orthogonal Projection Theorem.

Let $\mathcal{H}$ be a Hilbert space with the inner product $\langle \quad , \quad \rangle$ and T be a linear operator on $\mathcal{H}$.
6d. Prove that for every linear operator T on $\mathcal{H}$ there exists a unique linear operator T^* on $\mathcal{H}$ such that

$$\langle T\alpha, \beta \rangle \; = \langle \alpha, T^* \beta \rangle$$

for every α and β in $\mathcal{H}$. (We call such T^* an adjoint of T.)

Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ be Hilbert spaces. We write $\| \quad \|$ to denote the norm on each of these spaces. Consider linear operators $T : \mathcal{H}_1 \to \mathcal{H}_2$ and $S : \mathcal{H}_2 \to \mathcal{H}_3$ such that $ST = 0$. Let $S\alpha = 0$. Consider the operator $L = TT^* + S^*S : \mathcal{H}_2 \to \mathcal{H}_2$
7a. Suppose for some positive constant C,

$$\|f\|^2 \leq C(\|T^* f\|^2) + \|Sf\|^2)$$

for every f in the intersection of the domain of T^* with the domain of S. Prove that L is invertible, i.e. L has an inverse L^{-1}.

7b. With the same assumption and notation as in 7a, we write $G = L^{-1}$. Then we have the Hodge decomposition

$$\alpha = TT^* G\alpha + S^* SG\alpha \, .$$

(We call such G Green's operator.)

If we assume in 7b , $S\alpha = 0$, then it follows that $SG\alpha = 0$. If we put $u = T^* G\alpha$,

7c. Prove that u is the unique solution to

$$Tu = \alpha \, .$$

Recall that a real-valued function f on $[0,1]$ is said to be Hölder continuous of order α if there exists a constant C such that

$$|f(x) - f(y)| \leq C|x-y|^{\alpha}$$

for every $x, y \in [0,1]$. Define

$$\|f\|_{\alpha} = \max \left| f(x) + \sup \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} \right|$$

8. Prove that for $0 < \alpha \leq 1$, the set of functions with $\|f\|_{\alpha} \leq 1$ is a **compact** subset of the space $C[0,1]$ of real-valued continuous functions on $[0,1]$.

9a. State Hahn-Banach Theorem

9b. Use 9a to prove that Reisz Representation Theorem does not hold for Banach space $L^{\infty}[0,1]$, or the dual space of $L^{\infty}[0,1]$ is not $L^{1}[0,1]$.

10a. State Radon-Nikodym Theorem

10b. Let μ, ν, and λ be σ-finite. Show that if $\nu << \mu << \lambda$, then their Radon-Nikodym derivatives satisfy

$$\left[\frac{d\nu}{d\lambda}\right] = \left[\frac{d\nu}{d\mu}\right]\left[\frac{d\mu}{d\lambda}\right]$$

where $\nu << \mu$ denote ν is absolutely continuous with respect to μ.

11a. State the Fubini theorem.

11b. Let $X = Y = [0,1]$, $\mu =$ Lebegue measure on $[0,1]$, $\lambda =$ counting measure on Y. Let

$$f(x) = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{otherwise} \end{cases}$$

Does the Fubini theorem hold in this case? Justify your answer?

12a. Prove that the Lebesque measure of the Cantor set is zero.

For any set $S \subset \mathbb{R}$, we write $|S|$ for the diameter of S:

$$|S| := \sup\{|x-y| : x, y \in S\}.$$

If $|S| < \infty$ and $\alpha > 0$ we define the α-covered length of S as

$$H_{\alpha}(S) = \inf\left\{ \sum_{n=1}^{\infty} |C_n|^{\alpha} : S \subseteq \bigcup_{n=1}^{\infty} C_n \quad \text{where} \quad C_n \subset \mathbb{R} \right\}$$

The Hausdorff dimension of S is defined as

$$\dim_H(S) = inf\{\alpha > 0 : H_{\alpha}(S) = 0\}$$

12b. Prove that the Hausdorff dimension of the Cantor set is $\frac{\log 2}{\log 3}$

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