Aug 15, 2016 | TOPOLOGY QUALIFY EXAM | 2:00-5:00pm

Instructions: You have three hours to finish the exam. There are three parts to the exam. For the proofs, provide justifications and make your arguments clear, but try to avoid excess detail.

I

Definitions and statements of theorems

1. Define what it means for a topological space to be connected and locally-connected.
2. Define the compact-open topology on the space $\mathscr{C}(X,Y)$ of continuous functions from X to Y.
3. Let $\{U_1,\dots,U_n\}$ be a finite indexed open covering of the space X. Define what it means for an indexed family of continuous functions $\phi_i\colon X\to[0,1]$, $i=1,\dots,n$, to be a partition of unity dominated by $\{U_i\}$.
4. State the Tychnoff theorem.
5. Define what it means for a space to be contractible.
6. Define what it means for a covering space Y of X to be normal (or regular).
7. State the path lifting property for a covering space $p\colon Y\to X$.

II

Solve THREE of these problems.

1. Connected and path-connected spaces
   (a) Prove that the continuous image of a connected space is connected.
   (b) Show that if a space is path-connected, then it is connected.
   (c) Show that if a subspace $A\subset X$ is connected then so is its closure $\bar{A}$.
2. Compact and Hausdorff spaces
   (a) Show that if A and B are compact subspaces of a space X, then so is $A\cup B$. If in addition X is Hausdorff, show that $A\cap B$ is compact.
   (b) Show that a compact subspace of a Hausdorff space is closed, but that a compact subspace of a non-Hausdorff space need not be closed.
   (c) Let $\mathbb{R}_K$ denote the real numbers in the topology having as basis all open intervals (a,b) and all sets of the form $(a,b)\setminus K$, where $K=\{1/n|n\in Z_+\}$. Show that $\mathbb{R}_K$ is Hausdorff, but not regular.
3. Quotient spaces

(a) Prove that a surjective continuous open map $q\colon X \to Y$ is a quotient map.

(b) Find a quotient map that is not open.

(c) Show there exists a quotient map $T^2 \to S^2$, where S^2 is the 2-sphere and T^2 is the torus. (You do not need to give formulas for the map, a precise geometric description will be sufficient. Careful proofs of continuity are not required.)

4. Product and metric spaces

(a) Given a product space X = $\prod_{\alpha\in J} X_\alpha$ with the product topology, let $p_\alpha\colon X \to X_\alpha$ be projection onto the α-coordinate. If $f: A \to X$ is a function, show that f is continuous if and only if $p_\alpha \circ f\colon A \to X_\alpha$ is continuous for every $\alpha \in J$.

(b) Show that every compact metrizable space has a countable basis for its topology.

(c) Let (X, d) be a metric space whose diameter equals 4. Let $\mathscr{U}$ be the open cover $\{X - \{x\} \,|\, x \in X\}$. Prove that 3 is a Lebesgue number for $\mathscr{U}$.

5. Local properties

(a) True or False: The continuous image of a locally-connected space is locally-connected. If true, give a proof. If false, give a counter-example.

(b) Show that if X is locally-compact and Hausdorff then its one-point compactification $\overline{X}$ is Hausdorff.

(c) Let X be an uncountable set with the discrete topology. Prove that the one-point compactification $\overline{X}$ of X cannot be imbedded into the plane $\mathbb{R}^2$.

III

Solve THREE of these problems.

1. Fundamental group and applications

(a) Let S^2 be the 2-sphere and let S^1 be its equator. Prove that S^1 is not a retract of S^2.

(b) Show that if $A \subset X$ is a deformation retract. Let $a_0 \in A$. Show that the inclusion map $i\colon (A, a_0) \to (X, a_0)$ induces an isomorphism on fundamental group.

(c) Write down a presentation for the fundamental group of the wedge $T^2 \vee S^1$ of the torus and the circle.

2. Covering spaces and applications

(a) Suppose $p\colon Y \to X$ is a covering map. Show that if X is connected, then the sets $p^{-1}(x)$ have the same cardinality for all $x \in X$.

(b) Assume $p\colon Y \to X$ is a covering map and that X and Y are connected and locally-path connected. If $p_*\colon \pi_1(Y, y) \to \pi_1(X, x)$ is surjective what conclusion can be drawn about p?

(c) Let $\mathbb{P}^2$ be the projective plane. Show that any map $f\colon \mathbb{P}^2 \to S^1$ is null-homotopic.

3. CW complexes and applications of Van Kampen's theorem

   (a) Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point. Put a cell complex structure on X and use this to compute $\pi_1(X)$.

   (b) Show that the complement of finitely many points in $\mathbb{R}^3$ is simply-connected.

   (c) The suspension of a topological space X is the quotient space $\Sigma X = X \times [0,1]/\sim$ where $(x,t) \sim (y,s)$ if and only if either $(x,t) = (y,s)$ or $s = t = 1$ or $s = t = 0$. Prove that the suspension ΣX is simply connected if X is path-connected.

4. Covering spaces of the wedge of two circles. Consider the wedge of two circles $X = S^1 \vee S^1$ whose fundamental group $\pi_1(X, x_0)$ is the free group $F(a,b)$ on $\{a,b\}$. In each part below, you are asked to draw a covering space $p\colon (Y, y_0) \to (X, x_0)$ with the indicated properties.

   (a) An infinite sheeted covering space which is not normal.

   (b) An infinite sheeted covering space which is normal.

   (c) p is finite-sheeted and the subgroup $p_*(\pi_1(Y, y_0))$ contains the element a^3 but not the element aba.