Ph.D. Qualifying Exam in Analysis

January 12, 2018

There are 10 questions on this three-hour exam. Answer as many of them as you can.

1. Suppose f and g are continuous functions on $\mathbb{R}$ and $f = g$ almost everywhere on $\mathbb{R}$. Prove that $f = g$ everywhere on $\mathbb{R}$.

2. Suppose $E_1, E_2, E_3, \ldots$ are measurable subsets of $[0, 1]$ such that $E_m \cap E_n = \emptyset$ whenever $m \neq n$. Show that $\lim_{n\to\infty} m(E_n) = 0$.

3. Suppose F is a measurable set in $\mathbb{R}^n$ with finite measure, and suppose $\{E_k\}_{k\in\mathbb{N}}$ and E are Lebesgue measurable subsets of F. Show that if $\chi_{E_k}(x)$ converges pointwise to $\chi_E(x)$ on $\mathbb{R}^n$ as $k \to \infty$, then $m(E_k)$ converges to $m(E)$.

4. Suppose $\{f_n\}$ is a sequence of nonnegative measurable functions on a measurable set $E \subset \mathbb{R}$, and suppose that $\{f_n\}$ converges in measure to a function f on E. Show that
$$\int_E f\ dx \leq \liminf_{n\to\infty} \int_E f_n\ dx.$$

5. Show that
$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(x^n)}{x^n}\ dx = 1.$$
Justify all the steps in your answer. (Hint: consider the integrals over $[0, 1]$ and $[1, \infty)$ separately.)

6. Define $\pi : \mathbb{R}^2 \to \mathbb{R}$ by $\pi(x, y) = x$. Show that there exists a measurable subset $E \subseteq \mathbb{R}^2$ such that $\pi(E)$ is not measurable.

7. Using the Fubini/Tonelli theorems to justify all steps, evaluate the integral
$$\int_0^1 \int_y^1 x^{-3/2} \cos(\pi y/2x)\ dx\ dy.$$

8. Suppose μ and ν are finite positive measures on the measurable space (X, Σ). Show that there is a nonnegative measurable function f on X such that for all E in Σ,
$$\int_E (1 - f)\ d\mu = \int_E f\ d\nu.$$

9. Suppose $E \subset \mathbb{R}$ is a measurable set, with $0 < m(E) < \infty$, and $1 \leq q < r < \infty$. Prove that if f is a measurable function on E, then
$$\left(\frac{1}{m(E)} \int_E |f|^q\ dx\right)^{1/q} \leq \left(\frac{1}{m(E)} \int_E |f|^r\right)^{1/r}.$$

10. Let ℓ^2 be the Hilbert space of all sequences $x = (x_1, x_2, x_3, \ldots)$ such that $\sum_{i=1}^\infty |x_i|^2 < \infty$, with norm $\|x\| = \left(\sum_{i=1}^\infty |x_i|^2\right)^{1/2}$. Let ℓ_0^2 be the set of all $x = (x_1, x_2, x_3, \ldots)$ such that $x_i \neq 0$ for only finitely many i. Show that ℓ_0^2 is dense in ℓ^2.