Topology Qualifying Exam
August 16, 2019

PART I: Definitions/Examples *Thoroughly state each indicated definition and fully describe any examples that are required. In this section it is NOT required that you give proofs of your assertions. Work all five problems.*

1. For a topological space X define each of the three terms: connected, path connected, and simply connected. Clearly describe the relationships between the terms and provide examples showing that no two are equivalent.

2. State Urysohn's Lemma for a normal topological space. Given an example showing that the conclusion of the lemma can fail for a space that is not normal.

3. Define what it means for a continuous function $f : X \to Y$ to be a quotient map. Give two concrete examples of continuous surjections for which one is not a quotient map and the other is.

4. For a topological space $(X, \mathcal{T})$ define what it means for a collection of sets $\mathcal{B}$ to be a basis for $\mathcal{T}$. Then define what it means for a collection of sets $\mathcal{S}$ to be a subbasis for $\mathcal{T}$.

5. Let X be a topological space and $A \subseteq X$ a subspace. Define what it means for A to be a retract of X and what it means for A to be a deformation retract of X. Give three examples where (a) A is not a retract, (b) A is a retract but not a deformation retract, and (c) A is a deformation retract.

PART II *Work three of the problems from this section. Proofs should be based entirely on definitions and major theorems. Any theorems that you invoke should be carefully described and definitions should be clear from the context of your explanation.*

6. Show that a separable metric space is second countable.

7. Let N be a positive integer. Show that N is the number of path components of a topological space which is homotopy equivalent to a discrete set with N elements.

8. Show that for any set J the product space I^J is contractible. (Here I denotes the Euclidean interval $[0, 1]$ and I^J has the product topology.

9. (a) Show that a closed subset of a compact set is compact.
(b) Show that a compact Hausdorff space is regular.

PART III *Work three of the problems from this section. Explanation should be given for each assertion you make. Proofs should be based entirely on definitions and major theorems. Any theorems that you invoke should be clearly described and definitions should be clear from the context of your explanation.*

10. Give a statement and proof of the Unique Path Lifting Lemma for a covering map $p : \widetilde{X} \to X$.

11. In this problem $\mathbb{Z}_n$ indicates the cyclic group of order n.
(a) Show that the groups $\mathbb{Z}_2 * \mathbb{Z}_3$ and $\mathbb{Z}_3 * \mathbb{Z}_3$ are not isomorphic.
(b) Describe path connected topological spaces X and Y whose fundamental groups are isomorphic to the groups in (a).

12. Let X be a topological space containing elements x and y. Prove that $\pi_1(X, x)$ is isomorphic to $\pi_1(X, y)$ if there is a path from x to y in X.

13. Let X be the quotient space obtained from a closed 10-gon in the plane by identifying the 10 edges using the pattern $abcc\bar{b}\bar{a}ccc\bar{a}$.
(a) Show how to apply van Kampen's theorem to obtain a presentation for the fundamental group $\pi_1(X, x_0)$ where x_0 comes from a point in the interior of the 10-gon.
(b) Is the universal cover of X compact? Explain.

14. Let $X = S^1 \vee S^1$ be the wedge of two circles.
List all of the groups up to isomorphism that can arise as the covering transformation group for a six-sheeted connected covering space of X. For each one, give an example of a covering space of X which realizes that group. Clearly indicate which of your examples are regular coverings.