# ALGEBRA QUALIFYING EXAM - JANUARY 2023

## Instructions

Solve all the problems. Give clear and complete justification of your solutions, unless prompted otherwise for a particular problem.

## Problems

1. Let $G$ be a group of order $p^2$, where $p$ is prime. Show that $G$ is abelian.
2. Show that any group whose order is 385 has a non-trivial center.
3. Find the Galois group of $f(x) = x^4 + 5x^2 + 6 \in \mathbb{Q}[x]$; describe the automorphisms. Determine a familiar group that this Galois group is isomorphic to.
4. Let $F$ be a field of characteristic 0 and $\alpha$ the solution to an irreducible cubic in $F[x]$. Show that $F(\alpha^2) = F(\alpha)$.
5. For each of the following, either prove the statement or provide a counterexample.
   (a) Every matrix of finite order in $\mathrm{GL}(n, \mathbb{C})$ is diagonalizable over $\mathbb{C}$.
   (b) Every matrix of finite order in $\mathrm{GL}(n, \mathbb{Q})$ is diagonalizable over $\mathbb{Q}$.
6. Consider a polynomial $f(x)$ with integer coefficients. Assume $f(0)$ and $f(1)$ are odd. Show that $f$ has no integral roots.
   Hint: odd is the same as being congruent to 1 mod 2.
7. Consider the $S_3$ be the symmetric group on 3 letters and let $R$ be the group ring $R = \mathbb{Z}[S_3]$.
   (a) Write down a nonzero element in $R$ which is a zero-divisor.
   (b) Write down an element in the center of $R$ which is not in $\mathbb{Z} \cdot e$, where $e$ is the identity element of $S_3$.
   (c) Write down an $R$-module which is not a free module.
8. Classify up to similarity all linear transformations $T \in \mathrm{End}(\mathbb{C}_6)$ such that $T^6 = 0$ and $T$ has at most two 2-dimensional invariant subspaces.