TOPOLOGY QUALIFYING EXAM — JANUARY 2023

1. Definitions and examples

Please clearly state definitions, and describe your examples precisely. **Solve *all* problems in this section.**

Problem 1.1. Define what it means for a topological space to be *compact*.

Let $\mathbb{R}^2$ have the standard (euclidean) topology and let $\mathrm{pr}_1 : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x$ and $\mathrm{pr}_2 : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto y$ be the coordinate projection functions. Give an example of a subspace $C \subseteq \mathbb{R}^2$ which is not compact but each of the images $\mathrm{pr}_1(C) \subseteq \mathbb{R}$ and $\mathrm{pr}_2(C) \subseteq \mathbb{R}$ are compact.

Suppose D is a closed subspace of $\mathbb{R}^2$ and $\mathrm{pr}_1(D) \subseteq \mathbb{R}$ and $\mathrm{pr}_2(D) \subseteq \mathbb{R}$ are compact. What can you conclude about D?

Problem 1.2. Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Give the definitions of the *product topology* and the *box topology* on the product $\prod_{\alpha \in J} X_\alpha$.

Compare these two topologies on the countable product $\prod_{i=1}^{\infty} \mathbb{R}$ where $\mathbb{R}$ has the standard (euclidean) topology; say (with justification) if one is strictly finer than the other.

Problem 1.3. Define what it means for a topological space to be *connected*.

Suppose that $\mathbb{R}$ has the standard (euclidean) topology and that $X = \prod_{i=1}^{\infty} \mathbb{R}$ is given the box topology. Is X connected? Either supply a proof that X is connected or describe an explicit separation of X as necessary.

Problem 1.4. Given a continuous map $f : X \to Y$ between topological spaces, and $x_0 \in X$ a basepoint, define the *induced homomorphism* f_* between the appropriate fundamental groups.

Give an example of a pair of topological spaces X and Y and two continuous maps $f, g : X \to Y$ where f_* is injective and g_* is not injective.

Problem 1.5. Give the definition of a *basis* of a topological space X.

Describe two distinct bases for the standard (euclidean) topology on $\mathbb{R}^2$, and say how you would verify that these two bases generate the same topology on $\mathbb{R}^2$.

2. Point-set topology

Solve 3 of the following problems. In your answers, please clearly indicate what theorems you are using.

Problem 2.1 (Components). Let $X = \mathbb{R} - \mathbb{Q}$ and C be the (middle thirds) Cantor set. Recall that C can be defined as follows. Let $f : \mathbb{R} \to \mathbb{R} : x \mapsto x/3$ and $g : \mathbb{R} \to \mathbb{R} : x \mapsto x/3 + 2/3$. Set $C_0 = [0,1]$, $C_1 = f(C_0) \cup g(C_0)$ so the open middle third of C_0 is removed. Inductively, define $C_{n+1} = f(C_n) \cup g(C_n)$ for $n \geq 0$. The Cantor set is $C = \cap_{n=0}^{\infty} C_n$ with the subspace topology inherited from the standard topology on $\mathbb{R}$.

(a) Give the definition of a *connected component* of a topological space Z.
(b) Prove that the connected components of the space X above are singletons.
(c) Prove that the connected components of the Cantor set C above are singletons.
(d) Prove that X and C are not homeomorphic.

Problem 2.2 (Compactness). (a) Sketch a proof of the fact that a continuous bijection from a compact to a Hausdorff space is a homeomorphism. (State the major results used in your proof).
(b) Let X be the quotient space of the unit interval $[0,1]$ by the equivalence relation $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Prove that X is homeomorphic to the unit circle $S^1 \subseteq \mathbb{R}^2$.

Problem 2.3 (Hausdorff). (a) Prove that a topological space X is Hausdorff if and only if $\{(x,x) \mid x \in X\}$ is closed in $X \times X$ with the product topology.
(b) Prove that the product topology on $X \times Y$ is Hausdorff if and only if X and Y are Hausdorff.

Problem 2.4 (Point-set topology of covering maps). (a) Give the definition of a covering space map $p : E \to B$.
(b) Prove that a covering space map $p : E \to B$ is an open map. Conclude that p is also a quotient map.
(c) Is every quotient map $q : X \to Y$ a covering space map? Give a proof or a counterexample.

Problem 2.5 (Maps of Cantor sets). (a) Describe how to construct a continuous surjection from the Cantor set C to the unit square $[0,1] \times [0,1]$. For your reference, one definition of the Cantor set C is given in the preamble to problem 2.1 above.
(b) Sketch how to deduce that there exists a continuous surjection $[0,1] \to [0,1] \times [0,1]$

3. Fundamental group and covering spaces

Solve 3 of the following problems. In your answers, please clearly indicate what theorems you are using.

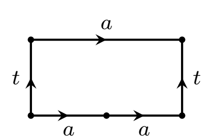
(The set $\mathbb{R}$ of real numbers is always endowed with the standard topology, $\mathbb{R}^n$ with the product topology, and subsets like $[0,1] \subset \mathbb{R}$, $S^1, D^2 \subset \mathbb{R}^2$, etc with the subspace topology.)

Problem 3.1 (Free groups and graphs)**.** (a) Let X be a connected, finite graph (1-dimensional cell complex). The fundamental group of X is a free group. Write down an expression for the rank of this free group in terms of the number of vertices (0-cells) and the number of edges (1-cells) of X.

(b) Let F be a free group of rank 2 and let $H \leq F$ be a subgroup of finite index, k. Using the theory of covering spaces, prove that H is a free group and determine its rank.

(c) Give a sketch of a topological proof of the fact that every subgroup of a free group is free. Just supply the major steps (no detailed arguments needed).

Problem 3.2 (Cell Complexes; Retracts)**.** Consider a cell complex X with 1-skeleton $S^1_a \vee S^1_t$ (one 0-cell v and two oriented 1-cells labeled a and t) and 2-cell shown.



(a) Write down a presentation for the fundamental group $\pi_1(X, v)$. (No need for any detailed justifications)

(b) Compute the abelianization of the group $\pi_1(X, v)$ obtained above.

(c) Is the subspace S^1_a a retract of X? Justify your answer.

(d) Same question as part (c) above for the subspace S^1_t.

Problem 3.3 (Covering spaces and fundamental groups)**.** Let $X \subseteq \mathbb{R}^3$ be the union of S^2 and one of its diameters.

(a) Use van Kampen's theorem to compute the fundamental group of X.

(b) Describe all the connected covering spaces of X. Say why your list is complete.

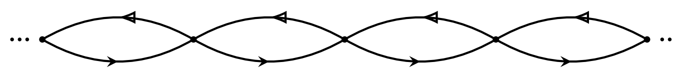
Problem 3.4 (General Lifting Theorem)**.** (a) State the general lifting theorem for continuous maps $f : (Y, y_0) \to (B, b_0)$ into the base space of a covering space $p : (E, e_0) \to (B, b_0)$. Note that this theorem involves topological conditions on the space Y as well as an algebra condition.

(b) Prove that every continuous map $f : \mathbb{R}\mathrm{P}^2 \to T^3$ is null-homotopic. Here $\mathbb{R}\mathrm{P}^2$ is the real projective plane and $T^3 = S^1 \times S^1 \times S^1$ is the 3-torus.

Problem 3.5 (Deck Transformations)**.** (a) Give the definition of a *deck transformation* (also called *covering space automorphism*) of the covering space $p : E \to B$, and sketch an argument that deck transformations form a group under composition. This group is denoted by $\mathrm{Aut}(E \xrightarrow{p} B)$.

(b) Let $B = S^1 \vee S^1$ be the wedge of two circles and $p : E \to B$ be a covering space. Prove that deck transformations must map 0-cells (resp. 1-cells) of E to 0-cells (resp. 1-cells) of E.

(c) Let $B = S^1 \vee S^1$ be the wedge of two circles as above. Determine the group $\mathrm{Aut}(E \xrightarrow{p} B)$ for each of the following connected, infinite-sheeted covering spaces. In each diagram the pattern continues infinitely to the right and to the left.

I. 

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