Qualifying exam Topology January 2024

Name: ....

(a) Please provide proofs for all your answers and claims;
(b) All subsets of $\mathbb{R}^n$ are given the subspace topology unless stated otherwise.

1. **(3+3 points)** Let $x_n \in X$ be a convergent sequence.
(a) Fix $y \in X$ such that $x_n \to y$. Is $Y = \{y\} \cup \{x_n : n \geq 1\}$ compact?
(b) Is $Z = \{z \in X : x_n \to z\} \cup \{x_n : n \geq 1\}$ compact?

2. **(2+2 points)** Consider the points
$$x^{(n)} = (0, 0, \ldots, 0, 1, 1, \ldots) \in X = \mathbb{R}^\omega$$
(n zeros initially, followed by ones only). Does the sequence $x^{(n)}$, $n \geq 1$, converge when X is given the: (a) product topology; (b) box topology?

3. **(3+2 points)** Let X be a T_1 space (= points are closed) whose topology has a basis consisting of sets that are both open and closed.
(a) Show that X is totally disconnected (= the connected components are one point sets).
(b) Give an example of such a space that is not discrete.

4. **(3+2 points)** (a) Consider the equivalence relation $0 \sim 1$ on $\mathbb{R}$ (more explicitly: $x \sim y \iff x = y$ or $x = 0, y = 1$ or $x = 1, y = 0$). Show that the quotient space $\mathbb{R}/{\sim}$ is not homeomorphic to $\mathbb{R}$.

   (b) Find a non-trivial equivalence relation $\sim$ on $\mathbb{R}$ (*non-trivial* means that there are $x \neq y$, $x \sim y$) for which $\mathbb{R}/{\sim} \cong \mathbb{R}$.

5. **(2+2+2 points)** Let $p : X \to Y$, $p(x) = y$, be a continuous map, and denote the induced homomorphism between the fundamental groups by $p_* : \pi_1(X, x) \to \pi_1(Y, y)$. True or false? Please give a proof or a counterexample:
(a) If p is injective, then so is p_*;
(b) If p is surjective, then so is p_*;
(c) If p is bijective, then so is p_*.

6. **(3+3 points)** (a) Show that a retract of a contractible space is contractible.

   (b) Deduce from the result of part (a) that $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$ is not a retract of $\overline{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

7. **(2+3 points)** Find the fundamental groups $\pi_1(X, x)$ of the following spaces. *Suggested method:* Relate these spaces to familiar ones by using the notion of a deformation retract.
   (a) a cylinder $X = C$, defined as the quotient of the square $Q = \{(x, y) \in \mathbb{R}^2 : -1 \leq x, y \leq 1\}$ by the equivalence relation $(-1, y) \sim (1, y)$;
   (b) a punctured cylinder $X = C \setminus \{p\}$, where p is the point represented by $(0, 0)$.

8. **(4 points)** Apply the Seifert-van Kampen theorem to find the fundamental group $\pi_1(X, x)$ of a square with diagonals. More formally, we can define X as the following subset of $\mathbb{R}^2$:

$$\begin{aligned} X =&\{(x,0) : 0 \leq x \leq 1\} \cup \{(1,y) : 0 \leq y \leq 1\} \cup \{(x,1) : 0 \leq x \leq 1\} \\ &\cup \{(0,y) : 0 \leq y \leq 1\} \cup \{(d,d) : 0 \leq d \leq 1\} \cup \{(d,1-d) : 0 \leq d \leq 1\} \end{aligned}$$

9. **(2+4 points)**(a) Let X be a compact space and suppose that $D \subseteq X$ is a closed set that becomes a discrete topological space with the subspace topology. Prove that D is finite.

   (b) Let $p : E \to B$ the universal cover of B, with E, B being path connected, locally path connected T_1 spaces. Assume that E is compact. Show that then $\pi_1(B, b)$ is finite.